Gaussian model

Ref: Kardar vol 2, ch4
Goldenfeld, ch 12.
Wilson & Kogut review

Recap of how we arrived at the model:

Starting from the Ising Hamiltonian $H = -\sum_{ij} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i$ with $\sigma_i = \pm 1$
and taking the Hubbard Stratonovich transformation, gives the partion in terms of a new set of continuous variables $\psi_i \in \mathbb{R}$.

$$\mathcal{Z}_N = \mathcal{N}\int \prod_{i=1}^{N} d\psi_i \; e^{-\sum_{ij}\psi_i J_{ij}\psi_j + \sum_i h_i\psi_i + \sum_i \log\cosh\left(2\sum_j J_{ij}\psi_j\right)}$$

Ising magnetization is related by

$$m_i = \frac{\partial}{\partial h_i}\mathcal{Z}_N$$

In the continuous limit $\psi_i \to \psi(\vec{r})$ and $J_{ij} \to J(\vec{r}-\vec{r}')$

$$\mathcal{Z}_N = \int \mathcal{D}[\psi]\, e^{-\mathcal{L}[\psi]}$$

With
$$\mathcal{L} = \int_a^L d\vec{r}\, d\vec{r}'\; \psi(r) J(\vec{r}-\vec{r}')\psi(\vec{r}') - \int_a^L dr\, h(r)\psi(\vec{r}) - \int_a^L dr\, \log\cosh\left(2\int dr' J(r-r')\psi(\vec{r}')\right)$$

For short-range interactions, using a Taylor expansion

$$\psi(\vec{r}') = \psi(\vec{r}) + (\vec{r}-\vec{r}')\nabla\psi(\vec{r}) + \frac{1}{2}\sum_\alpha\sum_\beta (\vec{r}-\vec{r}')_\alpha(\vec{r}-r')_\beta \nabla_\alpha\nabla_\beta\psi(\vec{r}) + \cdots$$

We write

$$\int d\vec{r}\int d\vec{r}'\, \psi(\vec{r}) J(r-\vec{r}')\psi(\vec{r}') \simeq \int d\vec{r}\cdot J_0\cdot\psi(\vec{r})^2 - \frac{1}{2}\cdot J_0\,\xi_0^2\cdot\int d\vec{r}\,(\nabla\psi(\vec{r}))^2 + \cdots$$

Where
$$\int d\vec{r}'\, J(\vec{r}-\vec{r}') = J_0$$
$$\int d\vec{r}'\,(\vec{r}-\vec{r}')^2 J(\vec{r}-\vec{r}') \simeq \xi_0^2 J_0$$
with ξ_0 being the range of interaction $J(\vec{r}-\vec{r}')$.

Similarly
$$\log\cosh\left(2\int d\vec{r}'\, J(\vec{r}-\vec{r}')\psi(\vec{r}')\right)$$
$$\simeq \log\cosh\left(2J_0\psi(\vec{r}) + J_0\xi_0^2\nabla^2\psi(\vec{r}) + \cdots\right)$$
$$\simeq 2J_0^2\psi(\vec{r})^2 + 2J_0^2\xi_0^2\,\psi(\vec{r})\nabla^2\psi(\vec{r}) + J_0^2\xi_0^4(\nabla^2\psi(\vec{r}))^2 - 8J_0^4\psi(\vec{r})^4 + \cdots$$

leads to a Landau-Ginzburg free energy

$$\mathcal{L} = \int d\vec{r}\left\{\frac{t}{2}(J_0)\,\psi(\vec{r})^2 + \frac{K_1}{2}(J_0)\,(\nabla\psi(\vec{r}))^2 + \frac{K_2(J_0)}{2}(\nabla^2\psi(\vec{r}))^2 + \frac{b(J_0)}{4}\psi(\vec{r})^4 + \cdots - h(\vec{r})\psi(\vec{r})\right\}$$

$$+\dots \qquad\qquad - h(\vec{r})\psi(\vec{r}) \Big\}$$

When we keep only terms upto quadratic order, it is the Gaussian model.

Remark: The case with t, K_1, b, and h non-zero, and rest of the term being zero is the Landau-Ginzburg Hamiltonian we discussed.

Remark: A similar $\mathcal{L}[m(\vec{r})]$ can be derived by coarse graining the Ising magnetization.

$$e^{-\mathcal{L}[m]} = \sum_{\sigma_i} e^{-\mathcal{H}[\{\sigma_i\}]} \prod_{j\in\vec{r}} \delta_{\sum_j \sigma_j, m(\vec{r})}$$

For mean-field Ising, this can be done exactly, even at single site level, and one gets

$$\mathcal{L} = -\sum_{ij} J_{ij} m_i m_j - \sum_i h_i m_i + \sum_i \left[\frac{1+m_i}{2}\log\frac{1+m_i}{2} + \frac{1-m_i}{2}\log\frac{1-m_i}{2}\right]$$

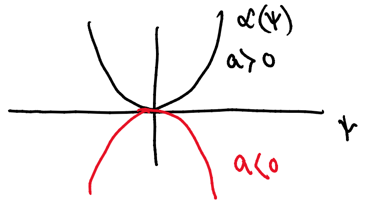
Exact analysis of the Gaussian model

The Gaussian model

$$\mathcal{L} = \int d\vec{r}\, \Big\{ \frac{t(J_0)}{2}(\psi(\vec{r}))^2 + \frac{K_1(J_0)}{2}(\nabla\psi)^2 + \frac{K_2(J_0)}{2}(\nabla^2\psi)^2 + \dots - h\psi(\vec{r}) \Big\}$$

Only ψ^2 order terms.

There is a phase transition when $t(J_0)$ changes sign, although the low temperature phase is unstable because of absence of a confining ϕ^4 term.



The Gaussian model is exactly solvable because the Fourier modes are non-interacting.

$$\hat{\psi}(\vec{q}) = \int d\vec{r}\, \psi(\vec{r})\, e^{i\vec{q}\cdot\vec{r}} \quad \text{and} \quad \psi(\vec{r}) = \frac{1}{L^d}\sum_{\vec{q}} e^{-i\vec{q}\cdot\vec{r}}\, \hat{\psi}(\vec{q})$$

for a finite system of volume L^d, the $\vec{q}$-space is discrete of infinitesimal boxes of size $\left(\frac{2\pi}{L}\right)^d$. Largest $\vec{q}$ value is limited by microscopic length scale $\vec{r}^{-1}$ (lattice unit) such that $\vec{q}$ are from a Brillouin zone.

For brevity, we shall denote

$$\frac{1}{L^d}\sum_{\vec{q}} \xrightarrow[L\to\infty]{} \frac{1}{(2\pi)^d}\hat{\int} d\vec{q}$$

[An useful identity, $\int d\vec{r}\, e^{i\vec{q}\cdot\vec{r}} = (2\pi)^d\,\delta(\vec{q})$ and $\int d\vec{q}\, e^{-i\vec{q}\cdot\vec{r}} = (2\pi)^d\,\delta(\vec{r})$]

This gives

$$\int d\vec{x}\ \psi(\vec{x})\psi(\vec{x}) = \frac{1}{(2\pi)^{2d}} \int^{\Lambda} d\vec{q}\, d\vec{q}'\ \psi(\vec{q})\psi(\vec{q}') \int d\vec{x}\ e^{-i(\vec{q}+\vec{q}')\cdot\vec{x}}$$

$$= \frac{1}{(2\pi)^d} \int^{\Lambda} d\vec{q}\ \psi(\vec{q})\psi(-\vec{q})$$

Similarly

$$\int d\vec{x}\ (\vec{\nabla}\psi)^2 = \frac{1}{(2\pi)^d} \int^{\Lambda} d\vec{q}\cdot q^2 \cdot \psi(\vec{q})\psi(-\vec{q})$$

Then, the Landau functional

$$\mathcal{L} = \frac{1}{(2\pi)^d} \int^{\Lambda} d\vec{q} \left\{ \frac{t + K_1 q^2 + K_2 q^4 + \dots}{2} \right\} \underbrace{\psi(\vec{q})\psi(-\vec{q})}_{|\psi(\vec{q})|^2} - h\psi(0)$$

using $\psi(-q) = \psi^*(q)$

Then, the q-modes are non-interacting and the partion function can be evaluated exactly.

[See Kardar, vol2, ch 4.6]

Remark: The exact solution shows that the free enenergy density is singular for $t(J_0)=0$, with the singular part

$$f_{na} \simeq -t^{d/2}\left[c_1 + \frac{h^2}{2t^{1+d/2}} \right] \qquad \text{for} \quad t>0$$

$$\simeq t^{2-\alpha}\, \Psi_f\left(h/t^{\Delta}\right) \qquad \text{with} \quad \alpha = 2-\frac{d}{2} \quad \text{and}\ \Delta = \frac{2+d}{4}.$$

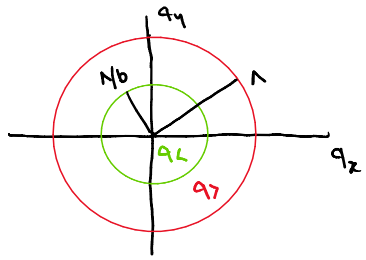
Notic how leading singularity do not involve the couplings K_2, and higher orders. They are irrelevant couplings in RG language. We shall see this in our RG calculation below.

RG analysis of the Gaussian model.

Starting point

$$\mathcal{Z} = \int \mathcal{D}[\hat{\psi}(\vec{q})]\ e^{-\frac{1}{(2\pi)^d}\int^{\Lambda} d\vec{q}\ \frac{t+K_1 q^2 + K_2 q^4 + \dots}{2} |\hat{\psi}(\vec{q})|^2 + h\hat{\psi}(0)}$$

Although Brillouin zone has a particular shape, we shall approximate it as a hypersphere. This is acceptable given that singularities are expected near $q \to 0$ modes.



RG steps:

(1) decimation: coarse-graining over fluctuation in real space in length scale $\Lambda^{-1} < x < b\Lambda^{-1}$ is equivalent to integrating over fourier modes in $\frac{\Lambda}{b} < q < \Lambda$. Denoting the fourier modes $q_< < \Lambda/b$ and $\frac{\Lambda}{b} < q_> < \Lambda$ we write

$$\mathcal{Z} = \int \mathcal{D}[\hat{\psi}(q_<)] \int \mathcal{D}[\hat{\psi}(q_>)]\, e^{-\mathcal{L}[q]}$$

Since, for Gaussian model all q modes are decoupled

$$\mathcal{L}[q] \equiv \mathcal{L}[q_<] + \mathcal{L}[q_>]$$

$$\Rightarrow \int \mathcal{D}[\hat{\psi}(q_>)]\, e^{-\mathcal{L}[q_>]} = \int \mathcal{D}[\hat{\psi}(q_>)]\, e^{-\frac{1}{(2\pi)^d}\int_{\Lambda/b}^{\Lambda} dq_>\, a(q_>)\, |\hat{\psi}(q_>)|^2}$$

$$= \prod_{K=1}^{n} \int d\psi\, e^{-\frac{1}{L^d}\cdot a_K \cdot \psi^2}$$ total n modes in the integration shell.

$$= \prod_{K=1}^{n} \left(\frac{\pi \cdot L^d}{a_K}\right)^{1/2}$$

$$= \exp\left(\frac{1}{2}\sum_{K=1}^{n} \log \frac{\pi L^d}{a_K}\right)$$

$$= \exp\left(\frac{nd}{2}\log L + \frac{L^d}{2}\cdot\frac{1}{L^d}\sum_q \log\frac{\pi}{a(q)}\right)$$

$$= \exp\left(\text{constant} + \frac{L^d}{2}\cdot\frac{1}{(2\pi)^d}\int_{\Lambda/b}^{\Lambda} dq_>\, \log\frac{\pi}{a(q_>)}\right)$$

Writing together,

$$\mathcal{Z} = \mathcal{N}\cdot e^{\frac{L^d}{2}\int_{\Lambda/b}^{\Lambda}\frac{dq}{(2\pi)^d}\log\frac{2\pi}{t+K_1q^2+K_2q^4+\dots}} \cdot \int \mathcal{D}[\hat{\psi}(q_<)]\, e^{-\mathcal{L}_<[\hat{\psi}(q_<)]}$$

(2) Rescale:

$$\mathcal{L}_<[m(q_<)] = \int_0^{\Lambda/b} \frac{d\bar{q}_<}{(2\pi)^d}\left\{\frac{t+K_1q_<^2+K_2q_<^4+\dots}{2}\right\}|\hat{\psi}(q_<)|^2 - h\hat{\psi}(0)$$

Defining $\bar{q}_< = \frac{\bar{q}}{b}$ (this is equivalent to $\vec{r}' = b\vec{r}$ in real space)

$$\Rightarrow \mathcal{L}_< = \int_0^{\Lambda} \frac{d\bar{q}}{(2\pi)^d} \cdot b^{-d} \cdot \left\{ \frac{t + K_1 b^{-2} q^2 + K_2 b^{-4} q^4 + \cdots}{2} \right\} |\tilde{\psi}(q)|^2 - h\tilde{\psi}(0)$$

(3) <u>Renormalize</u>: Here $\tilde{\psi}(q)$ is an integration variable. Define a new variable

$$\hat{\psi}(q) = b^{-\frac{d+2}{2}} \tilde{\psi}(q)$$

This choice is to keep the coupling for the $(\nabla\psi)^2$ term unchanged, which corresponds to keeping the amplitudes of fluctuations same (or bounded). Its like keeping the contrast of the coarse-grain picture same. Another way to see this is that this makes the $(t=0, h=0)$ as a critical fixed point which we know apriori from exact solution.

We get,

$$\mathcal{L}_< = \int_0^{\Lambda} \frac{d\bar{q}}{(2\pi)^d} \left\{ \frac{b^2 t + K_1 q^2 + K_2 b^{-2} q^4 + \cdots}{2} \right\} |\hat{\psi}(\bar{q})|^2 - h\, b^{\frac{d+2}{2}} \hat{\psi}(0)$$

This gives the partition function in terms of new Landau functional

$$\mathcal{Z} = \int \mathcal{D}[\hat{\psi}(q)]\, e^{-\mathcal{L}[\hat{\psi}; t, K_1, K_2, \cdots, h]}$$

$$= e^{-L^d f_0} \int \mathcal{D}[\hat{\psi}(q)]\, e^{-\mathcal{L}[\hat{\psi}; t', K_1', K_2', \cdots, h']}$$

<u>The RG flow</u>: the coupling constants change as

$$t' = b^2 t \qquad h' = b^{\frac{d+2}{2}} h$$

$$K_1' = K_1 \qquad K_2' = b^{-2} K_2$$

higher order couplings has higher power in $1/b$.

Then the anomalous dimensions

$$y_t = 2, \quad y_h = 1 + \frac{d}{2}, \quad y_1 = 0, \quad \underbrace{y_2 = -2, \cdots, y_\ell < 0}_{\text{irrelevant directions}}$$

[Note, no linear approximation near a fixed point is used. The above simple form hold everywhere on the coupling space]

<u>Critical exponents</u>: the $(t=0, h=0)$ is a fixed point. The non-analytic part of the free energy density

$$f_{na}(t,h) = b^{-d} f_{na}\left(b^{y_t} t, b^{y_h} h\right)$$
$$= t^{d/y_t} \Psi_f\left(\frac{h}{t^{\frac{y_h}{y_t}}}\right)$$

Following the standard scaling analysis (discussed in previous lectures) we get the critical exponents

$$\alpha = \frac{4-d}{2}, \quad \beta = \frac{d-2}{4}, \quad \gamma = 1, \quad \delta = \frac{d+2}{d-2}, \quad \eta = 0, \quad \nu = \frac{1}{2}$$

These exponents α, γ, η, and ν match with their results from saddle point approx. Only two (β and δ) do not match. Of course these two exponents are defined in the low temp phase $T<T_c$, which is ill defined for Gaussian model. However the reason is something deeper. Infact, we shall see that the φ^4-term, which stabilizes the low temp phase, is irrelevant for $d>4$ and mean-field exponents are exact. Then RG should reproduce the correct meanfield exponents. The reason is that even though φ^4 term is irrelevant above $d=4$, it is a dangerously irrelevant term.

Dangerously irrelevant coupling:

$$\mathcal{L}_4 = \int d\vec{x} \left\{ \frac{1}{2}(\nabla\varphi)^2 + \frac{t}{2}\varphi^2 + u\varphi^4 - h\varphi \right\}$$

Near the critical point of this model we expect

$$f_{na}(t,h,u) = b^{-d} f_{na}\left(b^{y_t} t, b^{y_h} h, b^{y_u} u\right)$$
$$= t^{d/y_t} \Psi_f\left(\frac{h}{t^{\Delta_h}}, \frac{u}{t^{\Delta_u}}\right) \quad \text{with } \Delta_h = \frac{y_h}{y_t}, \; \Delta_u = \frac{y_u}{y_t}$$

$\Rightarrow$ Order parameter

$$m = \left.\frac{\partial f}{\partial h}\right|_{h=0} = t^{\frac{d}{y_t} - \Delta_h} \Psi_m\left(\frac{u}{t^{\Delta_u}}\right)$$

Knowing that for $d>4$, coupling u is irrelevant, we set

$$m \sim t^{\frac{d}{y_t} - \Delta_h} \Psi_m(0) \sim t^{\beta} \quad \text{and got } \beta = \frac{d}{y_t} - \Delta_h$$
$$= \frac{d}{2} - \frac{1+d/2}{2}$$
$$= \frac{d-2}{4} \quad \text{(the wrong result)}$$

The error came from the assumption that $\Psi_m\left(\frac{u}{t^{\Delta_u}}\right)$ is analytic around $u=0$.

We can see that this is not correct from the mean field saddlepoint analysis wich gives

$$m = \sqrt{\frac{|-t|}{u}}$$

For large t, ie away from criticality we expect the mean field to be valid, therefore,

$$m \simeq t^{\frac{d}{y_t} - \Delta_h} \psi\left(\frac{u}{t^{\Delta u}}\right) \sim t^{\frac{d}{y_t} - \Delta_h} \left(\frac{u}{t^{\Delta u}}\right)^{-1/2}$$

$$\sim \left(\frac{t^{\Delta u - 2\Delta_h + \frac{2d}{y_t}}}{u}\right)^{1/2}$$

Consistency demands

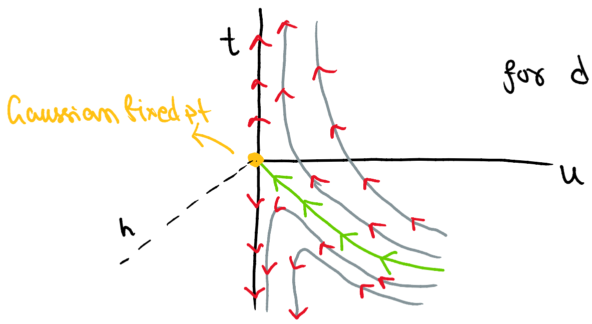
$$\Delta_u = 2\left(\frac{d}{y_t} - \Delta_h\right) = \frac{d-2}{2} \quad \text{and} \quad \psi(x) \sim x^{-1/2} \text{ for } x \to 0.$$

non-analytic.

Taking into account this non-analyticity gives the correct exponent $\beta = \frac{1}{2}$.

Bottom line, even though $u\varphi^4$ is an irrelevant term in RG for $d > 4$, it is a dangerously irrelevant variable.

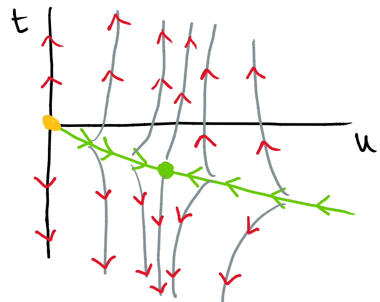
Remark: We shall see that $d=4$ is the upper critical dimension for $\mathcal{L}_4$. For $d>4$, there is a critical fixed point $(t=0, h=0, u=0)$ which gives the mean field critical exponents. This fixed point is the Gaussian fixed point.



for $d>4$.

For $d<4$, a new fixed point emerges close to the Gaussian fixed point, which makes the latter unstable.

for $d<4$.



This new fixed point is the Wilson-Fisher fixed point.

This means that u variable becomes a relevant variable for $d<4$, and it sends the RG flow towards the WF-fixed point which determines the critical exeponents

It is very hard to determine the flow around WF-fixed point. Fortunately, for $4-d$ small, the WF-fixed point appears very close to the Gaussian-fixed point and can be analyzed using perturbation theory (our next topic).